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Optimum Formulas for Buoyant Plume Rise

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III. MATHEMATICAL ANALYSIS OF CHIMNEY PLUME RISE AND DISPERSION

Optimum formulas for buoyant plume rise

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1. Introduction

While reliable plume rise formulas have long been needed for dispersion calculation, up to ten years ago there were very few data to test the half dozen or so published formulas. The literature is now becoming overpopulated with plume rise formulas, and, although many more data are now available, comparisons of formulas and data have not kept up. The author undertook a more comprehensive set of comparisons as part of a state-of-the-art report on plume rise for the Nuclear Safety Information Center of the Oak Ridge National Laboratory. Some of the results are summarized here.

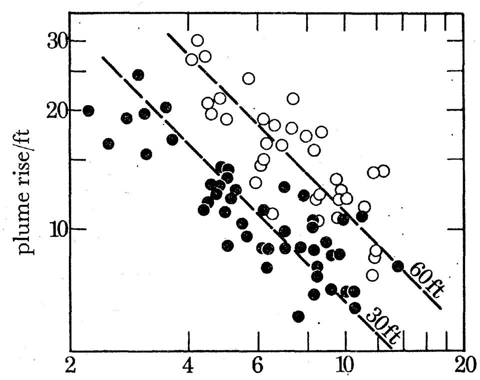
2. Near-neutral conditions

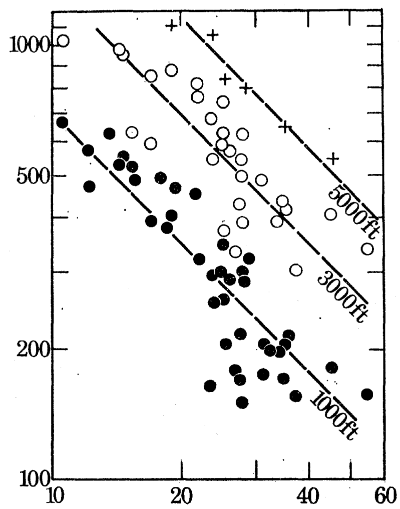
Data are relatively plentiful for near-neutral conditions, and a formula for this case is most often needed, particularly for determining the maximum ground concentration of a pollutant at the 'critical wind speed'. In selecting data for this case, the stability classification of each author was followed. Tennessee Valley Authority (T.V.A.) data, which had adequate temperature profile measurements, were rejected for this case if the air was unstable or if the plume was approaching its final height in stable air. Specifically, it was required that

$$x \leqslant 2us^{-\frac{1}{2}},$$

where x is the distance downwind, u is the mean wind speed at the plume level, and $s = (g/T)\,\partial\theta/\partial z$, where g is gravitational acceleration, T is the mean ambient absolute temperature, and $\partial\theta/\partial z$ is the mean potential temperature gradient from the top of the stack to the top of the plume (see figure 5). Altogether, observations from sixteen different sources were used for the near-neutral comparisons, including those of Ball (1958), Bosanquet (Priestley 1956) Harwell (Stewart Gale & Crooks 1958), Darmstadt and Duisburg (Rauch 1964), Tallawarra (Csanady 1961), Lakeview (Slawson 1966), Earley and Castle Donington (Lucas, Moore & Spurr 1963), Northfleet (Hamilton 1967), and six plants of the T.V.A. (Carpenter, Frizzola, Smith, Leavitt & Thomas 1967).

For each source, plume rise at one or several fixed distances was plotted against wind speed on logarithmic coordinates. There was a great deal of scatter in all of the plots, but in almost every case the points fitted well to a simple reciprocal wind speed law, as illustrated in figures 1 and 2. These are typical plots, one for the very small source of Ball and the other for the large steam plant of T.V.A. at Paradise, Kentucky. Only the plume rise at Duisberg was better fit by the $u^{-\frac{3}{4}}$ law of the recent Concawe (1967) formula; this formula itself was derived 75 % from the Duisburg data. Only the plume rise at the Shawnee plant of the T.V.A. showed a much stronger dependence on wind speed, possibly indicating some form of downwash. The $\Delta h \propto u^{-1}$ relation worked well in 12 of the 14 cases plotted, and is given by many theoretical and empirical formulas, so it seemed reasonable to accept it. Furthermore, it allowed considerable reduction of the data, since wind speed dependence could be eliminated by calculating the average value of Δhu for each source.





windspeed/ft s^{-1}

FIGURE 1. Plume rise at x = 30 and 60 ft (9 and 18 m) against wind speed for oil burner source of Ball. Heat emission = 0.04 MW.

FIGURE 2. Plume rise at x = 1000, 3000 and 5000 ft(Q_H) (305, 910 and 1520 m) against wind speed for Paradise steam plant of T.V.A. Heat emission = 90 MW.

The average product of plume rise and wind speed for each source was plotted against downwind distance on logarithmic coordinates, as shown in figure 3. The plume centre lines in general approximate a two-thirds power law, except for the Widows Creek and Johnsonville plants of T.V.A. (average slopes of 0.36 and 0.41). There is no indication of levelling in any of the observations, and it is obvious that the distance at which plume rise is measured is very important.

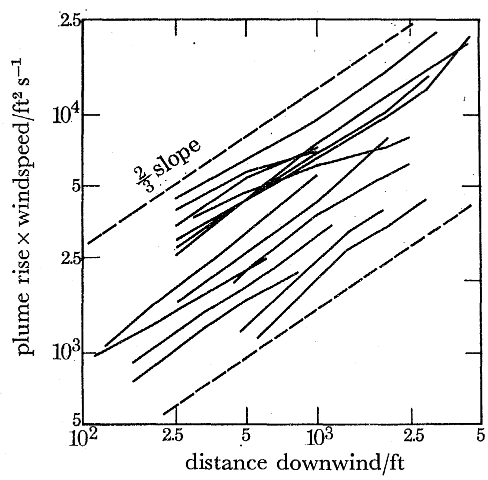


FIGURE 3. Average plume rise times wind speed against distance downwind. Reading top to bottom at x = 400 ft, sources are Lakeview, Johnsonville, Colbert, Widows Creek, Gallatin (one stack), Paradise (one stack), Paradise (two stacks), Tallawara, Gallatin (two stacks), Bosanquet, Shawnee, Duisburg, Darmstadt, Harwell system B, and Harwell system A.

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The average product of plume rise and wind speed at the greatest possible distance downwind for each source was calculated for comparison with formulas of the type $\Delta h \propto u^{-1}$. To assure adequate averaging, each calculation consisted of at least three periods of observations, with each period being $\frac{1}{2}$ to 2 h in duration. Two calculations were made for sources with two substantially different ranges of heat emission or with both one stack and two stack emissions. Also, two calculations were made for Harwell, where two different systems for measuring wind speed were employed.

This added up to 22 determinations of average plume rise times wind speed. It is felt that this large number and range of sources constitutes a better basis for comparison than ever used previously, and that the more equal weighting of data from different sources did much to reduce the influence of anomalies in certain sets of data. To reduce the anomalies further, a group of 12 determinations was chosen as 'selected data', eliminating the more questionable data. Reasons for elimination included suspected terrain downwash (Widows Creek and Northfleet), suspected lake shore influence (Tallawarra and Lakeview), very low heat emission (Ball 1958), more than 2 stacks operating (Colbert and Shawnee), unknown length of runs (Bosanquet, Carey & Halton 1950), and inadequate wind speed measurement (Harwell, system 'A'). Some *a posteriori* justification of these eliminations is that every formula tested gave better results with the selected data. The data are summarized in table 1, where h_s is the stack height, D is the stack diameter,

TABLE 1. DATA USED FOR COMPARISONS WITH NEAR-NEUTRAL FORMULAS

source	h_s/ft	D/ft	w/ft s^{-1}	Q_H/MW (stacks)	x/ft	$\overline{\Delta hu}$/ft^2 s^{-1}
Ball†	—	—	—	0.04	60	112
Harwell A†	200	11.3	32.8	4.6	2950	4430
Harwell B	200	11.3	32.8	4.6	1900	3980
Bosanquet†	?	6.5	31.9	6.4	600	2450
Darmstadt	246	7.5	15.7	3.6	820	2150
Duisburg	410	11.5	28.0	7.9	1150	3400
Tallawarra†	288	20.5	12.0	12.2	1000	5500
Lakeview†	493	19.5	65.0	48.5	3250	22100
C.E.G.B. plants						
Earley	250	12.0	18.3	6.4 (2)	4800	5580
Earley	250	12.0	56.0	19.7 (2)	4800	8150
Castle Don.	425	23.0	40.9	50.0 (2)	4800	14800
Castle Don.	425	23.0	54.7	67.0 (2)	4800	18600
Northfleet†	492	19.7	46.3	33.0 (2)	5900	10900
Northfleet†	492	19.7	70.0	50.0 (2)	5900	11150
T.V.A. plants						
Shawnee†	250	14.0	48.7	22.8 (8)	2500	6210
Colbert†	300	16.5	42.9	28.2 (3)	1000	7200
Johnsonville	400	14.0	94.8	45.2 (2)	2500	10100
Widows Creek†	500	20.8	71.5	70.2	2500	8000
Gallatin	500	25.0	52.4	70.6	3000	14250
Gallatin	500	25.0	23.7	35.8 (2)	2000	7850
Paradise	600	26.0	51.3	84.2	4500	21200
Paradise	600	26.0	57.2	91.6 (2)	4500	20000

† Not included in selected data.

w is the efflux velocity, and Q_H is the heat emission per stack. The number of stacks operating is shown for multiple stack cases. Note the large discrepancy between the observed values of plume rise at the Lakeview and Northfleet plants, in spite of the similarity of the two plants.

For each formula tested, the ratios of the calculated to observed values of plume rise times wind speed were computed. The median value of this ratio and average percentage deviation from the median were determined for all 22 data and also for the 12 selected data. The median value was used because it is less affected by anomalous data than is the mean value; the same is true of the average deviation (absolute value) as compared to the 'standard' deviation. The average deviation was expressed in terms of percentage of the median to give a better idea of the *relative* consistency of the formulas in predicting plume rise; the average percentage deviation is not affected by any readjustment of the coefficients in these formulas. The results of these calculations are shown in table 2. T_s is the absolute temperature of the stack gases and $\Delta T = T_s - T$, where T is the ambient value. The units are the same as those used in table 1.

TABLE 2. COMPARISONS OF NEAR-NEUTRAL FORMULAS WITH DATA

formula	$\Delta hu/\mathrm{ft^2\,s^{-1}}$	calc./obs. $\overline{\Delta hu}$: all data/%	calc./obs. $\overline{\Delta hu}$: selected data/%
Lucas I	$5700Q_H^{\frac{1}{4}}$	1.70 ± 91	1.46 ± 41
Lucas II (see Lucas 1967)	$2960Q_H^{\frac{1}{4}}(1+h_s/450)$	1.43 ± 61	1.30 ± 36
Moses & Carson	$885Q_H^{\frac{1}{2}}$	0.54 ± 34	0.48 ± 22
Ad hoc	$1000Q_H^{\frac{2}{3}}$	1.04 ± 24	0.97 ± 19
Stümke	$118D^{\frac{3}{2}}(\Delta T/T_s)^{\frac{1}{4}} + 1.5wD$	0.79 ± 27	0.73 ± 26
Oak Ridge, U.S.W.B.	$105Q_H + 1.5wD$	0.44 ± 37	0.46 ± 29
Priestley (see Csanady 1961)	$15.1Q_H^{\frac{1}{4}}x^{\frac{3}{4}}$	1.44 ± 27	1.44 ± 18
$\frac{2}{3}$ law	$18.3Q_H^{\frac{1}{3}}x^{\frac{2}{3}}$	1.16 ± 23	1.16 ± 12
modified $\frac{2}{3}$ law	above with max. rise at $x = 10h_s$	1.12 ± 17	1.14 ± 4

Of the established empirical formulas, the Stümke (1963) and the Moses & Carson (1968) ones work best, having the lowest percentage deviations. Better results are obtained with an *ad hoc* formula in which $\Delta hu \propto Q_H^{\frac{2}{3}}$. Since this formula was manufactured to fit the data, the author hopes that it will be ignored; it is almost axiomatic that, for every new selection of plume rise data and new method of analysis, there exists a new empirical formula that gives better results than all the old ones. The well-known '$\frac{2}{3}$ law' (Scorer 1959; Csanady 1961; Briggs 1965), in which the plume rise is proportional to $x^{\frac{2}{3}}$, is seen to give better agreement than any of the empirical formulas. This formula is based on the simplest possible theoretical model, in which a bent-over plume is emitted from a point source of conserved buoyancy and the plume radius grows in proportion to the height of rise. Still better results are obtained with a modified $\frac{2}{3}$ law in which the plume is assumed to stop rising at 10 stack heights downwind; an especially low average percentage deviation occurs with the selected data. Actually, very little data exist beyond 10 stack heights downwind, and one can hardly say that this assumption is verified. However, the modified $\frac{2}{3}$ law is strongly recommended for near-neutral conditions, because it is theoretically tenable, it includes a realistic dependence on distance downwind while setting a practical limit on plume rise, and most important, it gives far better agreement with the observations than any other formula in this analysis. Adjusting the coefficient for the best agreement with observations by dividing by 1.14, we have finally

$$\Delta h = 16Q_H^{\frac{1}{3}}u^{-1}x^{\frac{2}{3}} \quad \text{up to} \quad x = 10h_s,$$

where the units are feet, seconds and megawatts. In the original, dimensionally consistent form

$$\Delta h = 1.6F^{\frac{1}{3}}u^{-1}x^{\frac{2}{3}} \quad \text{up to} \quad x = 10h_s,$$

where

$$F = \frac{gQ_H}{\pi c_p \rho T} \doteq 1000Q_H \quad \mathrm{ft^4\,s^{-3}\,MW^{-1}}.$$

The data are plotted on dimensionless coordinates and compared to this formula in figure 4.

In the nine cases in which two stacks were operating, the average plume rise was slightly *less* than that of the single stack plumes, compared to the above formula (15 % less with all the data, 4 % less with only the selected data). This is probably not significant, except that the presence of a second plume apparently does not *enhance* plume rise in near-neutral conditions. At the Paradise plant, the addition of a second stack of about the same heat emission as the first resulted in almost no change in the average plume rise.

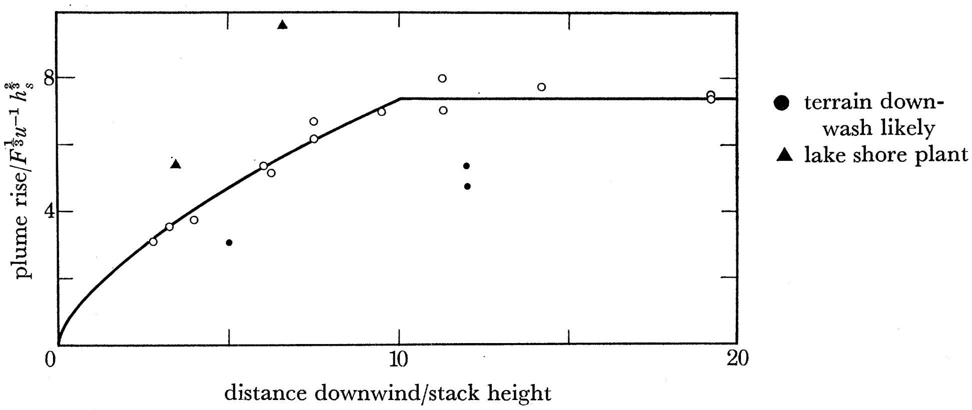

FIGURE 4. Non-dimensionalized average plume rise against number of stack heights downwind for data in table 1, compared to modified $\frac{2}{3}$ law. Plume centre line in neutral conditions at 12 plants (4–92 MW) with 1 or 2 stacks.

3. UNSTABLE CONDITIONS

There is only a small amount of data for conditions that are clearly unstable. On the basis of a total of eight periods, Slawson (1966) found just slightly higher rise in unstable than in near-neutral conditions. There was also more scatter when the atmosphere was unstable, no doubt due to increased convective turbulence. There also is a noticeable increase of scatter in the T.V.A. data for unstable conditions, but no significant change in the value of average plume rise times wind speed from the value for neutral conditions (calculations were made for Paradise and Gallatin plants with one stack operating, at $x = 3000$ ft (910 m)). It seems reasonable at this time to use the optimum formula for the near-neutral case for unstable conditions also.

4. STABLE CONDITIONS

There are also only a few data for stable conditions in which the temperature gradient is measured through the layer of plume rise, notably the T.V.A. data. The plume centre lines for the six periods in which single-stack plumes levelled off in stable are are plotted on non-dimensional coordinates in figure 5. The centre lines follow the two-thirds law at first, with about the same variation about the mean as in neutral conditions. The plumes bend over and reach a maximum rise at a distance $x = \pi u s^{-\frac{1}{2}}$, as predicted by most theoretical models. The average final height is given by

$$\Delta h = 2.9(F/us)^{\frac{1}{3}},$$

which is just a little higher than what is predicted by the simple model that leads to the two-thirds law, if a constant temperature gradient is assumed. The average deviation from this height is only 7 %, although the actual plume heights varied from 450 to 1500 ft (137 to 457 m).

The *top* of the stratified plume is given with equal accuracy by

$$\Delta h = 4.0(F/us)^{\frac{1}{3}}.$$

The plume rise increased by an average of 20 % over the above values when two stacks were operating at the T.V.A. plants, and by about 30 % at one plant with three stacks operating. The greatest enhancement occurred when the wind blew parallel to the line of stacks. The data are few, and since the mutual reinforcement of plumes depends on the closeness of the stacks, as well as on the wind direction, no modification of the single stack formula is recommended at this time.

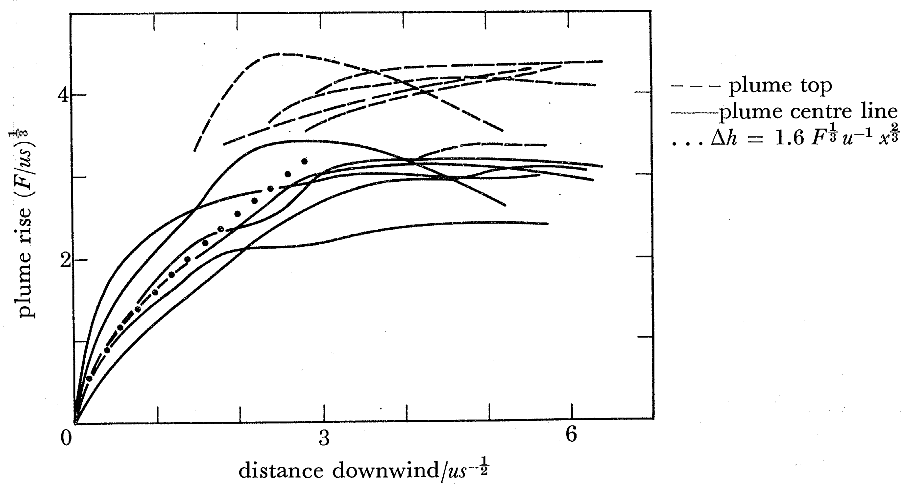

FIGURE 5. Non-dimensionalized plume centre line and plume top heights against non-dimensionalized distance downwind in stable conditions at Gallatin and Paradise plants, T.V.A.

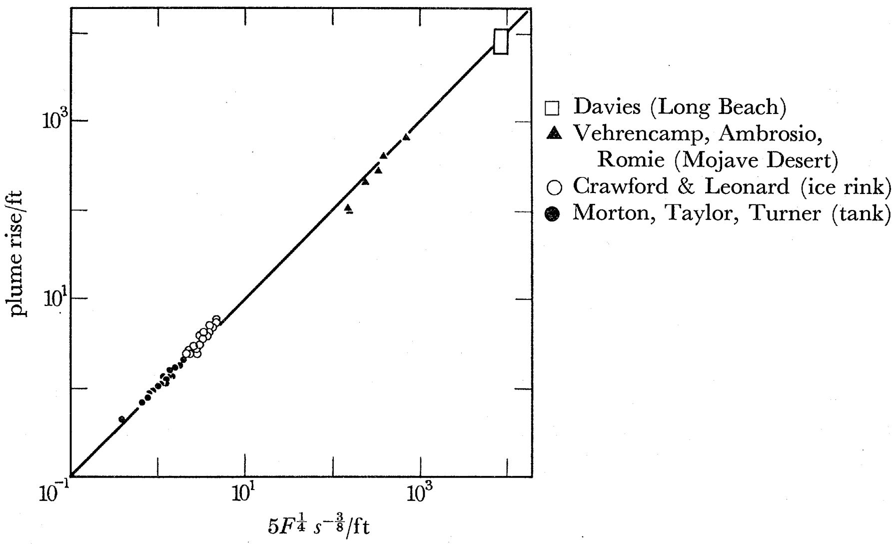

FIGURE 6. Plume rise against $5.0\, F^{\frac{1}{4}}\, s^{-\frac{3}{8}}$ for calm, stable conditions.

Finally, for buoyant plume rise in stable conditions that are nearly calm, the formula given by Morton, Taylor & Turner (1956) gives excellent agreement with a large range of data, as demonstrated in figure 6. Plotted are data from the modelling experiments in stratified salt solution of the above authors, the modelling experiment in an ice rink of Crawford & Leonard (1962), the oil fire experiments on the Mojave desert of Vehrencamp, Ambrosio & Romie (1955),

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and the estimates of the plume rise from a very large oil fire of Davies (1959; and private communication 1966). The formula recommended for the calm, stable case is

$$\Delta h = 5.0 F^{\frac{1}{4}} s^{-\frac{3}{8}}.$$

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